DOCUMENT RESUME

ED 466 735 SE 066 312

AUTHOR	Arzarello, Ferdinando
TITLE	Inside and Outside: Spaces, Times and Language in Proof Production.
PUB DATE	2000-07-00
NOTE	17p.; In: Proceedings of the Conference of the International Group for the Psychology of Mathematics Education (PME) (24th, Hiroshima, Japan, July 23-27, 2000), Volume 1; see ED 452 031. Based on joint research funded by the Italian Ministry of the University (MURST).
PUB TYPE	Reports - Evaluative (142) -- Speeches/Meeting Papers (150)
EDRS PRICE	EDRS Price MF01/PC01 Plus Postage.
DESCRIPTORS	Cognitive Structures; Elementary Secondary Education; Knowledge Representation; Learning Strategies; *Mathematics Instruction; Mathematics Skills; *Numeracy; *Proof (Mathematics); *Semiotics

ABSTRACT

This paper focuses on some cognitive and didactical phenomena that feature processes and products of pupils in grades 7-12 who learned mathematical proof within technological environments. The main issues in the analysis of students' performances consist of metaphors, deictics, mental times, narratives, functions of dragging, abductions, and multivariate language to be used within an embodied cognition perspective. The learning of proof is described as a long process of interiorization through the specific and complex mental dynamics of pupils. (Contains 46 references.) (DDR)

INSIDE AND OUTSIDE: SPACES, TIMES AND LANGUAGE IN PROOF PRODUCTION

Ferdinando Arzarello (°)
Dipartimento di Matematica, Università di Torino, Italia

Abstract. *The paper focuses on some cognitive and didactical phenomena which feature processes and products of pupils (grades 7-12), who learn 'mathematical proof' within technological environments. Language and Time reveal crucial and assume specific features when subjects interact with artefacts and instruments, because of the semiotic mediation by precise interventions of the teacher. The main issues in the analysis of students' performances consist in metaphors, deictics, mental times, narratives, functions of dragging, abductions, linear vs. multivariate language and so on, to be used within an embodied cognition perspective. The learning of proof is described as a long process of interiorisation, through specific and complex mental dynamics of pupils, from perceptions and actions within technological environments towards structured abstract mathematical objects, embedded in a theoretical framework.*

Introduction

The purpose of this paper is to focus on the genesis of (abstract) mathematical objects within specific mathematical areas, when pupils interact with technological artefacts and instruments (for the distinction see: Bartolini Bussi & Mariotti, 1999a), in particular the computer. The term 'mathematical object' is used here in a wide sense, which covers *concepts* (including representations: see Vergnaud, 1990) as well as *relationships* among them: we call *structured mathematical object*[1] such a cluster and use the abbreviation SMO to denote it. An example of SMO is the set of natural numbers (represented, let us say, in base ten): it is a set equipped with the function of successor, the usual operations and their properties.

More specifically, the paper sketches a theoretical framework, where some major variables in the genesis of SMOs are described and scrutinised; the model is based on the analysis of processes and products in pupils (aged 12 to 18) who approach 'mathematical proof' in the classroom[2]. In some cases, students work within

(°) The results are based on a joint research, funded by the Italian Ministry of the University (MURST), which involves many Italian researchers (all their presentations at the last PME's have been obtained within such a project) and that I co-ordinate. In particular the following people have contributed to the theoretical elaboration as well as to the experimental work: V.Andriano, P.Boero, M.G.Bartolini Bussi, G.Gallino, R.Garuti, M.A.Mariotti, M.Maschietto, F.Olivero, D.Paola, O.Robutti. A special acknowledgement is due to P.Boero, M.G.Bartolini Bussi, F.Olivero, D.Paola and O.Robutti for their useful criticism and suggestions in the writing of the manuscript.

[1] SMOs have some similarity with the way mathematical entities are presented in category theory, namely as objects equipped with arrows and represented through diagrams.

[2] In the whole, about 100 students were involved in our research: they have been observed and video-taped while exploring situations, conjecturing hypotheses, proving properties (generally in peer or group interaction); moreover their written protocols (sometimes individual, sometimes not) were studied. The pupils belong to various schools in different parts of Northern Italy, where the research-teachers of our project work. Generally, pupils of 14-18 years attend the Liceo Scientifico (that is a high school with a scientific option), whilst the youngest (12-13 y.) attend the last years of the junior secondary school.

computer-based environments (e.g. Cabri-Géomètre) or interact with other artefacts (e.g. Mathematical Mechanisms), in other cases they simply use paper and pencil; the mathematical areas exploited are Geometry and (elementary) Number Theory.

The genesis of SMOs will be pictured as a dynamic evolution from *perceptions*, *deictics*[3], *actions* to *mathematical objects, symbols, relationships* through a complex setting of *transformational processes*, which involve two main tools or categories:

(*i*) *language* (in its different forms: body l., oral l., written l.), by means of which subjects start, develop and support the whole genetic process;

(*ii*) *time*, as a psychological category which represents the mental environment where the genetic process of SMOs 'is born' and 'lives'.

The paper is divided into three sections: in *Section 1* the theoretical framework is discussed; in *Section 2* time and language are analysed with respect to the mental processes of pupils who are constructing SMOs; in *Section 3* an emblematic case study is discussed; some provisional conclusions are sketched at the end of the paper.

1. The theoretical framework

The nature and construction of mathematical objects is a main topic in Mathematics Education (Harel & Tall, 1991; Sfard, 1994, 1998). A fascinating point concerns the "question of the primary sources of our understanding" (Sfard, ibid., p.45), of the genesis of mathematical concepts and of their relationships with pupils' pre- and extra-school experience. A complete investigation of the problem is beyond the purpose of this paper. What we are interested in is the *embodied cognition* perspective (see: Johnson, 1987; Lakoff & Johnson, 1980; Lakoff & Núñez, 1996; Thurston, 1994) with particular respect to the following issues:

- *metaphors* as grounding and producing basic mathematics concepts (Lakoff & Johnson, 1980; Sfard, 1994; Radford, 1999, 2000);
- *mind times*, as mental environments where the subjects put and transform the experienced facts (Boero et al., 1996; Guala & Boero, 1999);
- *narratives*, through which subjects make meanings for what they have previously experienced by building up mathematical stories (Love, 1995; Mason & Heal, 1995; Nemirovski, 1996; Scrivener, 1995).

We shall now elaborate the sense that such notions have within our framework.

The issue of *embodiment*, that is "to put the body back into the mind" (Johnson, 1987, p.xvi), assumes in our research a particular flavour, insofar a mathematical proof may seem very far from our bodily experience, and discovering how a typically conceptual activity, as proving is, has its roots in "perceptual, motor-program, emotional, historical, social and linguistic experience" (Johnson, ibid.), reveals very

[3] The deictic function of language (see Radford, 1999, 2000) allows to indicate directly in the discourse some object which has not a name: words like "this", "that" are typically exploiting a deictic function.

intriguing. Embodiment and proof become particularly intermingled when pupils interact with an *artefact*, e.g. a mechanism, a computer with a dynamic geometry software (Bartolini Bussi, 1993; Mariotti & Bartolini Bussi, 1998, Bartolini Bussi et al., 1999c), or work within *fields of experience* (Boero et al., 1995b), e.g. the field of shadows (Boero et al., 1995c) or the field of natural numbers, conceived as chains built up from counting processes (Gallistel & Gelman, 1992). In such cases language and time reveal crucial, respectively as a tool and as a mental environment, which support the genetic process from perceptions towards SMOs and mathematical proof.

In fact, proving[4] is an activity where discursive and semiotic processes are essential, while the formalistic aspects are not so important (see Arzarello, 2000):

> "to expose, or to find, a proof people certainly argue, in various ways, discursive or pictorial, possibly resorting to rhetorical expedients, with all the resources of conversation, but with a special aim ... that of letting the interlocutor see a certain pattern, a series of links connecting chunks of knowledge" (Lolli, 1999, quoted in Arzarello, 2000).

At this point, mathematical embodiment knocks on the door. But how does this effectively enter the game? We shall argue that language and time are the two right ingredients to look at in order to grasp the genetic processes of pupils towards proof. The approach we shall develop takes into account the semiotic analysis of generalisation processes given by Radford (Radford, 1999, 2000). In a detailed study of novices' performances who, working in group and interacting each other, try to generalise and write in algebraic form regularities that are discovered in so called 'figural numbers', Radford points out that the transition to the abstract general algebraic formula is trigged and supported by two main functions of language, deeply intermingled with the metaphoric function:

(i) the *deictic* function (see note 3);

(ii) the *generative action function* (which supplies the conceptual dimension for generalising: see also the notion of *grounding metaphor for functions* in Lakoff & Núñez, 1996).

According to Radford's analysis, the two functions start and support the genesis of SMOs in algebra (in our terminology): language produces surrogates for (not yet existing) mathematical objects, which are grounded in the subjects' knowledge and fields of experience; metaphors are the tools by which subjects express this link and start creating that conceptual dimension, which will reveal essential for the construction of the mathematical object self.

Our claim is that deictic and generative action functions are present and important also in the geometric context, for example when pupils explore situations and formulate conjectures using Cabri or Mathematical Mechanisms. The way things

[4] We use the word *proof* referring both to proof as a final (usually written) *product* and to the proving *process* (see Douek, 1999); the meaning we refer to each time will be clear from the context.

happen is specific of the artefact. In particular, the generative action of (some types of) dragging reveals crucial (Arzarello et al., 1998b) with Cabri.

Moreover, observations of interaction and of dragging in peers working at the same computer with only one mouse show that a major component of pupils' behaviours and dynamics is time. This component appears also in other contexts: research carried out by Boero and his co-workers (Boero et al. 1996; Guala & Boero, 1999) has shown the relevance of mental times to analyse the components of mental dynamics in pupils working in different contexts. Guala & Boero introduce some categories for mental time: t. of past experience, contemporaneity t., exploration t., synchronous connection t.. All of them are more or less guided by the image of an *order* in a continuum of events (or in more continua), which can be transformed or seen in different ways by the subject. Guala & Boero's categories for mental time are important also in our study, especially for analysing the processes of generation of conditionality.

However, our research has shown also other aspects after which mental time enters into the genesis of SMOs, namely *'tempos'*[5], with consequent problems of *synchronisation*: see 2.2 for examples and comments. 'Tempos' and orders are crucial for the genetic process of SMOs and it will be shown that an artefact like Cabri seems to facilitate their activation [6].

2. Language and Time

Language is a crucial tool through which pupils, possibly with the support of teachers, elaborate their *daily experience* towards more sophisticated behaviours: from expressions describing everyday life (narratives: see below) to sentences which organise the experienced relationships into *causal, final, hypothetical* moves. Because of the verbal coaching, in these processes students' mental times are structured according to a double polarity: *the past*, that is the lived experiences recalled by memory; and *the future*, namely the space of anticipation and volition. In this sense, language and time are essential ingredients in the genesis of *theoretical knowledge*, that is "a system of scientific concepts" in the sense of Vygotsky (1934, chap. VI); such a knowledge is *a-timed*, hence essential transformations are necessary to construct it from experiences which are embedded in time. The evolution of pupils towards it requires a long period of apprenticeship in school and long-term interventions of teachers; developing a theoretical model for describing such an

[5] Other possible words to use are *rhythms* (which have a more external connotation) or *moves* (as actions in a sequence, with different quickness). For an elaboration on such concepts, see the vol. 879 of the *Annals of the New York Academy of Sciences*, 'Tempos in Science and Nature: Structures, Relations, and Complexity', in particular Varela (1999).

[6] It is interesting to observe that 'tempos' are important also in number contexts: in fact they often are the root of fields of experiences for pupils, who acquire the notion of number also through processes of counting down and up; pupils and teachers of elementary schools use the metaphor of the *chain* to describe them (Boero, 1995a). For another example see Bartolini Bussi et al. (1999).

evolution is one of the goals of our research but at the moment it is not yet completely elaborated. Specifically, we are studying the evolution of pupils' *solving abilities* in different contexts and the conditions which seem crucial for causing and supporting such an evolution, including the consequent didactical engineering.

For the clarity of exposition and due to space constraints, we shall limit to illustrate a condensed segment of such an evolution, namely we shall show the genesis of SMO's in students who have already undergone the process of apprenticeship towards theoretical knowledge but who pass again through its main steps in the emblematic example which we shall comment.

We shall start our illustration discussing with more details the analysis of language and time as tools for interpreting the genetic process of SMOs.

2.1 Analysis of Language

A first point to stress is the mixed production of non verbal and oral language (e.g. gestures of pupils who draw geometric figures in the air or on the screen, together with their oral comments): it possibly creates and always supports the partial, generally disconnected, order which is given to the facts experienced by the subjects within their mental time. But an order can exist since the subject can make sense of her/his experiences. At this point *narratives* enter the scene, in different forms. Nemirovsky (1996) defined a mathematical narrative[7] as a narrative articulated with mathematical symbols. This type of narratives is produced for ex. by pupils who reconstruct their past experience with Cabri or the Mathematical Mechanisms while they are moving to the proving phase (see Douek, 1999, for examples in different contexts). But there is also another type of narratives in mathematics, that has been studied by Mason & Heal (1995) and Scrivener (1995), namely the *sketches* (drawn on paper, but also made from gestures in the air). In fact the sketch may be a tool which activates the narrative function, insofar it is made by pure imagination (Scrivener, 1995). Goldschmidt (1991) distinguishes two modalities of visual reasoning while working with sketches: "seeing that" and "seeing as"; there is a dialectic between the two: "a back and forth swaying movement which helps translate particulars of form into generic qualities, and generic rules into specific appearances" (quoted in Scrivener). Hence its functions enter the dialectic between perceptual and conceptual, emphasized by many scholars (Laborde, 1999), and is deeply intermingled with verbal competencies and mental times.

A second issue concerns the deep link between language and pupils' mental dynamics. Mental dynamics 'give life' to perceptions, insofar they can be used to activate the past experience as well as to anticipate a future intention. In such cases,

[7] Ricoeur, who is often quoted by Nemirovsky, points out that one of the narrative's major functions consists in ordering different times: "the activity of narrating does not consist simply in adding episodes to one another; it also constructs meaningful totalities out of scattered events. The art of narrating...requires that we are able to extract a configuration from a succession" (quoted in Nemirovsky, 1996, p.198).

argumentative activity, especially in the interaction among mates (see section 3), has a double function: (i) scaffolding the construction of new (inter- and intra-conceptual) relationships; (ii) controlling and managing the whole (conjecturing, proving, etc.) process. The two functions may be integrated particularly when the discovered relationships, conjectures and proofs allow students to gain new insight in a mathematical problem or field. The link between language and pupils' mental dynamics is particularly interesting within the Cabri environment: very often, the *perception* of objects drawn in Cabri leads students to use *metaphors* and *deictics* to name them; immediately they start actions on them, which modify both the perceptual and the metaphoric aspects (*transformational* function: see Simon, 1996). The interactions among perceptions, metaphors, actions are the starting point of an *evolution* towards the structured mathematical objects.

2.2 Analysis of mental timed frames for 'tempos' and for orders

The evolution from perceptions towards SMOs generally happens within rich *timed frames*, which are ruled by the language and the actions of the subjects. A first frame is nurtured by different **'tempos'** which can be observed, e.g. working with Cabri. A first 'tempo' (present also in paper and pencil environments) consists of the periodic change from *ascending* to *descending control* of the subject with respect to the geometrical figures and backwards (see Saada-Robert, 1989; Arzarello, 1998a): it varies during the performance and marks also the change in the way students see the mathematical objects, with respect to what is considered as given and what is to be found (see section 3). It has low frequencies (in the first 20 minutes of the video we have counted about 15 such changes): for this reason, it is called a *slow 'tempo'*.

Another kind of 'tempos', which are faster, are present in the interaction between pupils and the software Cabri. Their roots seem to consist in the rupture of the synchronism which exists between thought and physical movements of our body (see the description in Berthoz, 1997, and Varela, 1999), when subjects use the mouse in a dynamic geometry software. The phenomenon is typical of novices but is present also in subjects who have more experience with the software (#61-64, #128)[8]; an example is given by group interactions, when the mouse is in the hand of one subject and the others cannot follow what happens on the screen (#92 and before). Generally the subject tries to overcome the gap between the two 'tempos', controlling hand gestures and dragging slowly and carefully so that the synchronism between perceptual and cognitive aspects can be established (#77, #102). Such 'tempos' are faster (Berthoz, 1997; Varela, 1999), hence they are called *fast 'tempos'*.

'Tempos' may support a continuity and correspondence between perception and thought, back and forth from inside to outside (see the genesis of circumcentre in the Conclusion). It seems that a feeble co-ordination of fast 'tempos' is an obstacle for solving problems using dynamic geometric software. On the contrary, the students

[8] #n indicates the sentence numbered n in the Appendix.

who master Cabri have a good co-ordination of fast 'tempos' in that context and this has positive consequences at the cognitive level also for the control of slow 'tempos', hence for the whole process of problem solving in Cabri. Moreover, it seems that some functions of the dynamic software (specifically, some types of dragging) are useful for generating a strong evolution from perceptions towards SMOs. In this sense, a conscious use of dragging (which requires a high co-ordination of fast 'tempos') can support the subject in the processes of generating generalities. This has radical consequences for the teaching: promoting a conscious use of dragging can be achieved only by suitable interventions of the teacher, who scaffolds the experiences of pupils and teach them the different typologies of dragging (the method is systematically used in the classrooms of our project).

In the case of numbers, the 'tempos' of counting can support the natural order of numbers and become an essential constituent of their field of experience: the search for a regularity starts with metaphors which try to build a new order from the facts experienced through numerical explorations within such a field and evolve towards suitable SMOs (see Boero et al., 1995a).

Language encompasses also the problems of the **order** of events, which appear not linearly ordered in perceptual experiences (Varela, 1999). Here the narrative function of language plays an essential role: it allows students to make sense of what has happened, connecting the perceptual experiences with the past and categorised ones. In other words, while metaphors and deictics allow the construction of objects in a discursive form and the beginning of a generalisation process which anticipates the *future*, the narrative function allows events to be ordered by looking mainly at the *past* (see: #104, #138 , #151, #152). The result of the evolution may be a *de-timed* sentence expressed in the *present tense*[9], which has the features of a scientific discourse.

Mental time becomes a sort of mental environment where subjects use language to order in some way the facts that they have experienced (see Varela, 1999). For ex., gestures and the broken oral language in #73 are an attempt to grasp the complex relationships among geometrical objects represented by Cabri figures; that is, subjects try to express discursively a non linearly ordered factual situation (i.e., Cabri figures which change because of dragging) with a tool (oral language) which is more comfortable with linear order[10]. This creates a tension which is expressed in the example by the broken sentences and utterances, as if language tried to mimic the complex order of the experienced facts.

An interesting point to stress is the fact that the variety of languages used by students in most of their performances helps communication. Illuminating examples are given by # 65, 66, 67 and # 73, 74: the verbal part of the message apparently has

[9] The language we refer to is Italian: the sense of its present tense is possibly different from English.
[10] See Simone, 2000, for a discussion on linear and non-linear languages.

no meaning; on the contrary, the pupils successfully communicate some essential point to their mates. The non-linear structure of her sentences reflects the different nature of what she is communicating: this can happen when all the 'tempos' have been synchronised among the girls and the perceptual and metaphorical ground of what has happened on the screen is shared among the three pupils. Consequently, the non linear order constitutes a common basis upon which they can share some ideas about the SMOs they are building: we call *multivariate* such a cluster of languages used together. The timed frame for orders constitutes a ground for the *cognitive unity* discussed by Garuti et al. (1996, 1998): in fact students can use a multivariate language to describe (#78) and communicate (#73) their ideas. Only after this multivariate language has done its job, namely has guaranteed a shared experience for the three girls, they start a further evolution process (namely from #127 to #152) towards a linear and more formal language, which explains in more canonical forms why what they have experienced is such (#150). The evolution from Perceptions-Deictics-Actions towards Mathematical Objects-Symbols-Relationships (that is SMOs) described till now can be sketched like in fig.1.

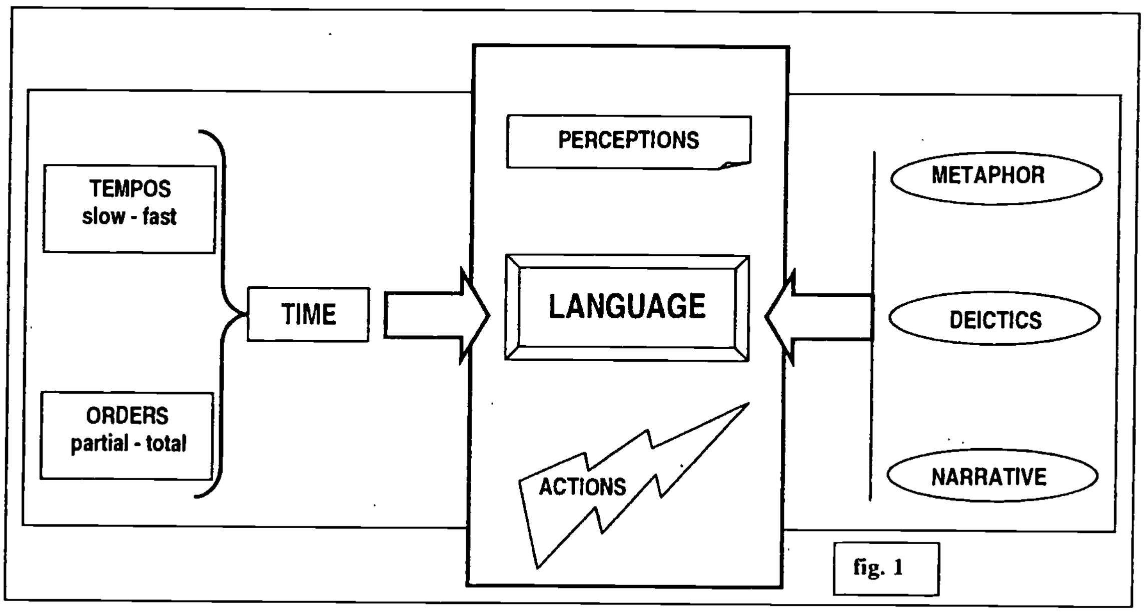

fig. 1

2.3 Other theoretical tools

Even if time and language are the two major ingredients of the model, the analysis of SMOs requires also many other tools; more precisely:

- the theory of *experience fields* (Boero et al., 1995b);
- the *discussion as a polyphony of voices* (Bartolini Bussi, 1996), typically when one voice is that of the official scientific knowledge and the others those of pupils;
- the notion of *semiotic mediation*, from the Vygotskyan historical-cultural school of psychology (Wertsch, 1998; Bartolini Bussi & Mariotti, 1999a, 1999b);
- *typologies of dragging* analysed by Arzarello and Olivero (Arzarello et al., 1998b).

Moreover, a new tool is in course of elaboration, namely the analysis of the *neurological bases* of some mathematical concepts (Berthoz, 1997; Dehaene et al., 1993, 2000; Longo, 1997): particularly the analysis of *perception* as a multisensorial integration, and of *action* as anticipation of movement given by Berthoz (1997) and discussed by Longo (1997) from the mathematical point of view.

Space constraints do not even allow us to sketch the issues mentioned above: the interested reader is invited to read the quoted reference for more information. The next section will use all the ingredients previously outlined to describe the genesis of SMOs in a concrete emblematic example.

3. A case analysis

Following up these general comments, we shall now give a more detailed analysis of the protocol in Appendix. In the sequel, H will indicate the dynamic figure ABCD and T the figure A'B'C'D'; in the protocol, the way pupils look at the two figures changes over time: sometimes they look at T as depending on H (we write $H \Rightarrow T$), sometimes students look for hypotheses on H so that T satisfies particular conditions (we shall write $T \Leftarrow H$). The different directions through which pupils look at the objects are deeply intermingled with their mental times, e.g.: order from past to present or from future to present, etc.; 'tempos' depending on the control with respect to the objects (ascending vs. descending), to the dialectic with dragging and sketches and to the outside-inside dynamics and back. 'Tempos', orders, causal and conditional dependencies find here a crucial connection: in particular the different orders that subjects attach to experienced facts may change or not in the transition from exploring to conjecturing and to proving and this may have consequences on the degree of difficulties found by pupils (see also Douek, 1999).

The starting point of the cognitive dynamic is #37, where M makes explicit the heuristics to be used, which is developed in the following items (till #48): this is the genesis of $H \Rightarrow T$, which lasts till #60. In #50 the degenerate point, which will be named in #58 begins to be present as a perceptual fact. The episode marks the beginning of a systematic exploration (descending control and attention towards the future). During the exploration (#52) the theory crops up from the past: together with the perceived object it produces the sentence #53, which is spelled in the de-timed present tense of standard scientific sentences. In #58-60 the genesis of SMOs starts: the point (of intersection of the perpendicular bisectors, when it exists) is manipulated through dragging and its meaning is framed into conditional sentences; that is the perceptual facts and the generalising function interact to generate a mathematical sense for what is perceived (#60). This produces an inversion in the way the objects are looked at ($T \Leftarrow H$), which culminates in #64; similarly the control becomes ascending, as it is shown by the *wandering dragging* (we use the analysis of dragging developed by Arzarello et al., 1998b) pursued in #61-64. There are also different 'tempos' (#61-67) in the students: E explores with Cabri (#62, 64: *wandering dragging*); V wants to think with her own 'tempo', which is different from E's (#65):

in fact she does not grasp the explanation of E (#60); M tries to synchronise the two (#66). In #68 E draws a parallelogram, which marks a new inversion (H $\Rightarrow$ T, descending control). The tension between perceptual and theoretic aspects is always high (#72): in #73 the multivariate language, which collects perceptual and theoretical chunks, allows V to synchronise her 'tempo' with that of E and to communicate with M (#74); namely, multi-variety and non-linearity of language allow communication and sharing of knowledge among the students (see the comments in 2.2). In fact, in #75 E transforms the sentence of V into a "linear" sentence, structured in more a canonical way, and in #76 V echoes the voice of E. In #77 there is a new inversion (T $\Leftarrow$ H, ascending control): the 'tempo' of E has changed and it seems it is now shared in the group. The slow 'tempos' of her gestures and the new modality of dragging (the so called *bounded dragging*) allow E and her mates to rule the guided exploration, which now starts from an already structured object. In this sense, the action with the mouse and the consequent dragging, with its complex typologies, can incorporate a deictic and a generalising function.

In #78-84 there is a systematic genesis of structured objects (different types of quadrilaterals): also Boero et al. (1996) observed this type of exploration in other contexts. The cases of the square and the rectangle are object of a mental experiment: E makes a mental dynamic exploration, namely she activates a strategy which is similar (but not identical) to the "changing hypothesis" strategy described by Boero (ibid.). The mental experiment produces a deduction in the form of impossible examples. In #82-91 exploration continues: there are many inversions of control and of the relationships between H and T; #92 is interesting: E makes a mental exploration supported by her hands' gestures (T $\Leftarrow$ H), which is transformed into "linear" language in #94. After the new 'impossible example' of #95 (ascending control), the degeneration is grasped at a new (more theoretical) level, which is marked by the linguistic transformations of #96, #100, #101, #102, #103. The last sentence expresses an *abduction* (Arzarello et al., 1998a), which is produced after the *lieu muet dragging* (Arzarello et al., 1998b) in #102, where B describes a circle. The linguistic transformations mark the abduction while the effects of dragging suggest the theory of remarkable points of a triangle, insofar they are linked to inscribed and circumscribed circles. Such a theory is evoked by E and the time is now towards the past (#104). In #104-126 the students give voice to the theories and discuss them; the resolving point is #126, where there is the genesis of the right figure. In #127 E gives *the* explanation in a multivariate sentence, in which however the theoretical side evoked in previous interventions (#104-125) is marked by the present (de-timed) tense. It is interesting to observe that in this present time do live both words and deictics but there is no reference to the particular figure on the screen in that moment: in fact the argument is supported by a sketch made on the screen by hands, which refers to a *general* geometrical object (Harel & Tall, 1991). Namely, because of the narrative incorporated in the sketch, thinking can go beyond the concrete perceptual aspects and get the theoretical side (they do not see "that" but "as", see 2.1). In fact, in #131 the "multivariate" sentence of #127 is transformed into an almost linear utterance, which incorporates also some chunks of the logical relationships among its constituent sentences.

Conclusions

The comment above illustrates how embodied is the generation of SMOs and how strong the cognitive unity from the first empirical perceptions to the production of logically structured sentences of mathematics can be. This is achieved insofar the students have interiorised the dragging practice through a cognitive apprenticeship (Arzarello et al., 1993), where the role of teacher is basic.

The written proof produced later on by E, V and M exhibits a strong continuity through linguistic transformations and (modulo some inversions between H and T) from all the explorations above to the final product, which has the canonical structure of usual mathematical proofs (something similar is described in Boero et al., 1996). Constraints of space do not allow us to describe it here. We limit ourselves to sketch the structure of the whole process in fig.2, where the main transformations are shown, namely: the genesis of conditional statements through *de-timing*; that of mathematical sentences through *linearisation* and that of abstract concepts (and their symbolic representation) as a *generalising transformation* of metaphors, deictics and narratives.

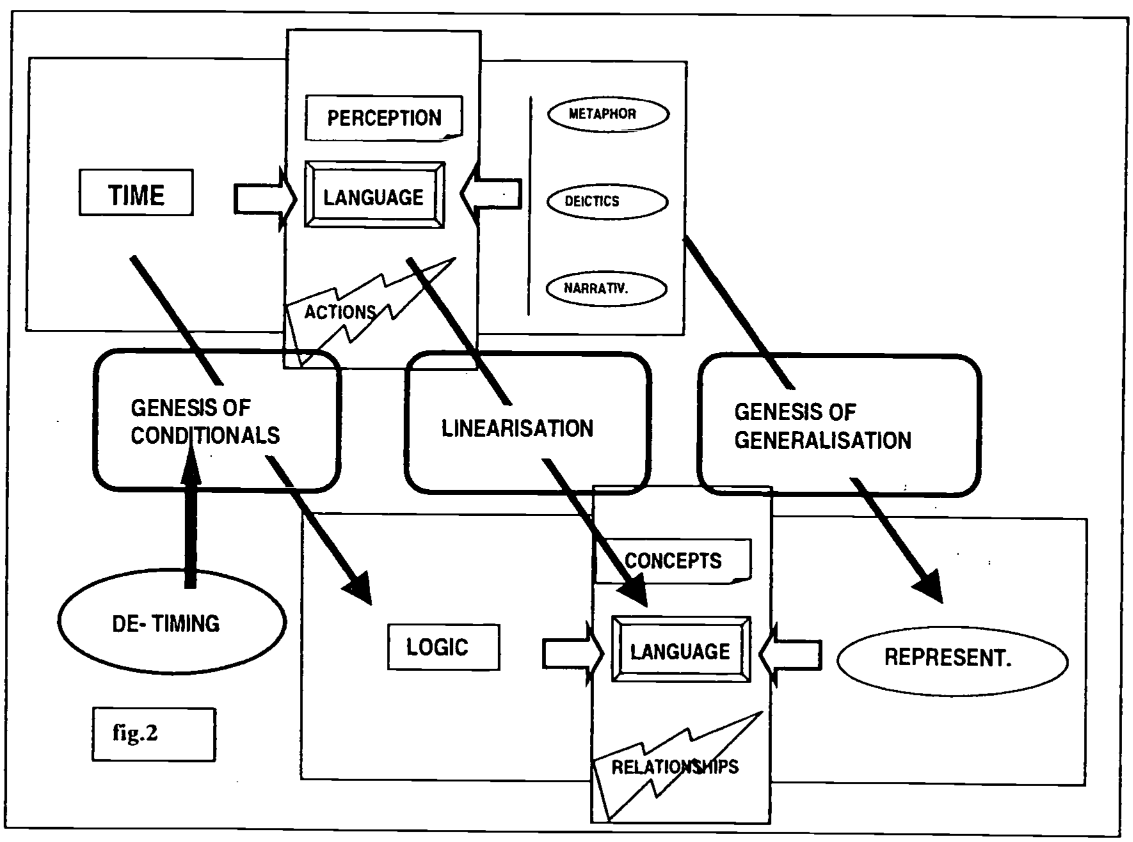

fig.2

Now, let us summarise the main points. We have illustrated the generation of SMOs in geometry within the embodied cognition perspective, analysing processes of pupils working with Cabri. Mental times of students revealed crucial in managing such a generation. Another crucial point was the mediation of the artefact, in particular the role of dragging and of the representations of mathematical objects (like Cabri drawings, sketches); the dynamic geometric figures have proved a *cognitive pivot*: the hypotheses incorporated in them are transformed into new ones through the complex dynamic of action and interpretation within changing tempos and orders of the subjects when interacting. An emblematic example is the story of the "degenerate point". At the beginning (#50) it is a purely perceptual fact; then it is a metaphor which describes the result of a dynamic process (#58), but whose meaning remains to be explained (#65); the first chunk of theoretical explanation is obtained (#72) through the framing of the perceptual experience in a narrative which encompasses different perceptions (#66 and ff.). In the end (#96) the word "degenerate" marks the conclusion of a rich exploration framed within a time section in the sense of Boero et al. (1996). Now things are ready for the final transformation, namely the "degenerate point" to be transformed in the "circumcentre" (#102, #127), whose genesis is the dragging described in #100 through the metaphor of "keeping the point inside".

Appendix: an emblematic protocol.

PROBLEM. *You are given a quadrilateral* ABCD. *Construct the perpendicular bisectors of its sides:* a *of* AB, b *of* BC, c *of* CD, d *of* DA. A' *is the intersection point of* a *and* b, B' *of* b *and* c, C' *of* c *and* d, D' *of* a *and* d. *Investigate how* A'B'C'D' *changes in relation to* ABCD. *Prove your conjectures.* [17-18 year old pupils, who know Cabri1 very well and are acquainted to explore situations when presented with open problems). We present some excerpts of the protocol of Elisa, Michela and Valentina's work with Cabri; the protocol is the transcription of a video, which lasts one and a half hour. We present some (parts of) episodes of the first part (which lasts 20 minutes altogether), where the genesis of SMOs is particularly evident. The three students work with one computer and have also paper and pencil at their disposal].

The first 4 minutes of the video, with 35 interventions of pupils, are skipped: the students read the problem, draw the figures, accomplish the constructions and give names to the created objects.

The students (E, M, V) constructed fig. 1 and

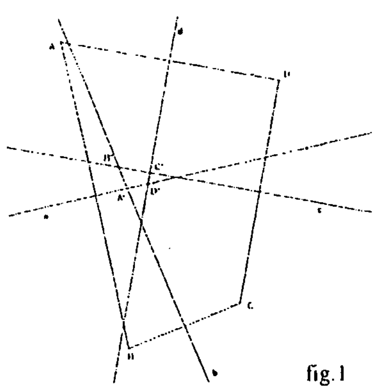

fig.1

checked its correctness through dragging.

36. E: and now? (E has the mouse)

37. M: One must see how it varies, as the external quadrilateral changes [ABCD]

...

41. V: I think that not...try moving...the figure...[E drags randomly point D] ... 'cause.... move this one [V indicates point B and E drags it randomly]...it seemed to me that you had put the...you know...the function of the segment, that you can create without doing the points...it seemed that you had not shot this one [A']...do you understand?

.....

(time = 5' 25")

48. V:...but if you already do it coloured...

you get a small coloured point.
[E colours the quadrilateral A'B'C'D' and drags the point D]
49. E. And let's try perhaps...let's try to see what happens with regular external quadrilaterals...
50 M: I don't know...let's start with a square, so that we see...
[E drags B, C, D up to get a rectangle]
51. V: Properties of the perpendicular bisectors? [the Italian word is *assi*]
52. E: The perpendicular bisector... how was it?
53. M: Hence...the perpendicular bisector passes through the midpoint...
54. E: It is perpendicular!...
55. M: ok!
56. E: Well in the square, in the...
[She measures the sides of ABCD, then drags randomly first point C then point B]
(time = 7' 10")
57. E: No, I was wondering...that...I was wondering!
[E stops the dragging with fig. 2]

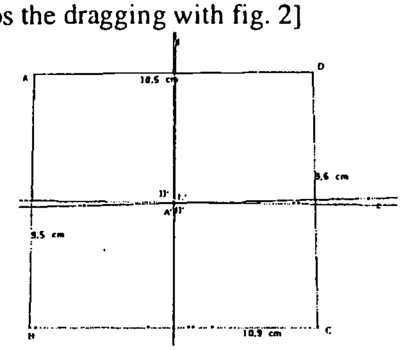

fig.2

58. E: No, that is...it degenerates into a point...it's logical isn't it?...if they are parallel...that is, if the sides are perpendicular...[she drags B]...
59. M: ...we are looking for...
60. E: I mean if the opposite sides are parallel [she continues dragging B], those [the perp. bisectors] are perpendicular. And up to this...Isn't it?....and if they are equal the midpoint is on the same line. [She drags C till ABCD becomes a rectangle].
61. M: ok...so?
[E drags A randomly, then D; in the end she goes back to the original figure]
(time = 9')
62. E: Please, tell me something!
[E drags B, D, C, A systematically]
63. V: What are you doing? Are you moving randomly?
64. E: No, I was wondering if I could construct a figure...
65. V: Listen to me, please; let's try thinking... just a moment...'cause of that we have done before...to finish the discourse, when it degenerates into a point, that...have I misunderstood or we have not explained it?
66. M: well, practically she is saying: since the properties of the perpendicular bisector are perpendicularity and the distance from a point...if...the different segments are parallel, then since they are perpendicular....Moreover if two of...like in a square for example, the midpoint must belong to the same straight line.
67. V: yes
[In the meanwhile E has dragged the points A, B, C, D in order to get a parallelogram]
68. E: I am doing a parallelogram...the sides are parallel, aren't they? in the parallelogram. Hence also the perpendicular bisectors are parallel, isn't it? They are parallel two by two.
69. V&M: yes
70. E: So also the segments A'B' and C'D' are parallel.
71. V: Hence it maintains...no, nothing!
(time = 11')
[E drags the points B, C, D till she gets a rectangle]
72. E: hence the square has been proved...degenerate...
73. V: Hence if...when...and hmm, yes, that is natural, because when there are two...the two sides of the external one...the two sides parallel two by two, it is natural...that is it should always be that the perpendicular bisectors are...
74. M: it is so.
75. E: Because they are parallel...they are perpendicular to two parallel lines.
76. V:...they are parallel...
77. E: let's move the point very slowly to see what changes [she drags the point C for a while]. Now they are not any longer parallel, hence...these two [d, b] are not any longer parallel...sure, it is logic...and not for these two [a, c] ...That is what we have said up now.
[E drags slowly point C along line BC and back]

78. E: Nothing, there is no way to get...the...that they are parallel...a square... type: inside there is a square; the sides should be parallel two by two...However one longer than the other! Isn't it? ...But it cannot ever be, because otherwise...then these [the sides of ABCD] are not any longer parallel.
79. M: It cannot be a square inside!
80. E: Sure!
81. M: Neither a rectangle!
82. E: It can be only...a trapezium [she drags C trying to get a trapezium]
83. M: Or a parallelogram.
84. E: A parallelogram or a trapezium.
85. V: I am wondering why...
...*(time = 14')*
[M takes the mouse and drags B]
92. E: No! Why the lines...should be so [she mimics with her hands two parallel horizontal lines]; then it means that one is longer than the other, isn't it?
93. V: sure! necessarily!
94. E: However if one is longer than the other, the other two are not parallel any longer...Otherwise in the parallelogram they have moved , since they all are parallel.
[M drags B and stops when the perpendicular bisectors coincide in one point].
95. V: Hence it can never be..
96. E: My God! It degenerates into a point again!
97. E: Excuse me, can I...? [she asks to move the mouse again and drags C].
98: Let us see what remains equal, when it remains...
99: V&M: when?
100. E: That is, I try dragging the stuff....and...the point...keeping the point inside...that is moving the point, but leaving that...that...the quadrilateral degenerates into a point.
101. M: That is, it keeps inside...
(time = 16' 02")
102. E: ...to find a property that...when it degenerates into a point...do you understand? [she drags the point B slowly, trying to keep the perpendicular bisectors coincide]
103. E: Let's mark the angles...excuse me, but in a triangle the intersection point of the perpendicular bisectors is...
104. V: hmm, I had already thought of that!
...[the students recall what they remember about the remarkable points of a triangle; they are in doubt whether to consider the circumcentre or the centroid; they make some exploration and discuss about the circles outside and inside the quadrilateral]...
126. M: That is when you can put a circle inside.
127. E: No, no! I know it! It is the circumcentre..., why it must be equidistant from the sides, isn't it? [she indicates with fingers on the screen] This point [the meeting point of the bisectors] is the perpendicular bisector of this [AD] hence it is equidistant [from A and D].
(time = 18' 13")
128. V: Wait a moment, stop please!
129. Hence this point here [the supposed circumcentre]...if you have...look at..., that is the bisector, it is equidistant from the extremes...
130. V: sure!
131. E: ...because the bisector is the locus of points which are equidistant from the extremes...hence it is equidistant from this and from this [A and B]. But from this [A], these two are equal...[she repeats the same reasoning and gestures with respect to all vertices]...hence in the end they are all equal and it is the ray, isn't it?...
132. V&M: yes!
133. E: ...I mean I wasn't that clear.
134. V: Try to do a circle.
135. M: Try!
136. E: hmm...how can I do it?
137. M: excuse me...centre and point; make the centre here [the intersection point of bisectors]...
138. E: Wait! No! It is enough to see...the angles. How are they when...it is a cyclic quadrilateral...the opposite...are 180°? [they ask the teacher the right property and then measure the angles in Cabri; E measures angles D and B; some explorations and reasoning concerning the sum of opposite angles D and B]
...*(time = 20' 43")*
150. E:....yes, in conclusion....hmm, anyway the proof is that...the perpendicular bisectors are equidistant from vertices...hence this point [the intersection point of the bisectors] is

equidistant from these two vertices [A, B] and also from these [A, D], from these [D,C] and from these [C,B]...isn't it?...then in the end ...Isn't similar to a problem we have already done?

151. V: We certainly did something similar!

152. E: perhaps with the medians.

REFERENCES

Arzarello F.; Chiappini G.P.; Lemut E.; Malara N. & Pellerey M.: 1993, 'Learning to program as a cognitive apprenticeship through conflicts', in Lemut et al. (eds.), *Cognitive Models and Intelligent Environment for Learning Programming,* NATO ASI Series, vol. F111, Berlin: Springer Verlag, 284-298.

Arzarello, F.; Micheletti, C.; Olivero, F.; Paola, D. & Robutti, O.: 1998a, 'A model for analysing the transition to formal proof in geometry', *Proc. PME XXII,* Stellenbosch, vol. 2, 24-31.

Arzarello, F.; Gallino, G.; Micheletti, C., Olivero, F., Paola, D. & Robutti, O.: 1998b, 'Dragging in Cabri and modalities of transition from conjectures to proofs in geometry', *Proc. PME XXII,* Stellenbosch, vol. 2, 32-39.

Arzarello, F.: 2000, 'The proof in the 20th century: from Hilbert to automatic theorem proving', in Boero P. (Ed.), *Theorems in School from History and Epistemology to Cognitive and Educational Issues*, Dordrecht: Kluwer Academic Publishers, to appear.

Bartolini Bussi, M.G.: 1993, 'Geometrical Proof and Mathematical Machines: An Exploratory Study', *Proc. PME-XVII,* Tsukuba, vol. 2, 97-104.

Bartolini Bussi M. G.: 1996,'Mathematical Discussion and Perspective Drawing in Primary School', *Educational Studies in Mathematics*, 31 (1-2), 11-41.

Bartolini Bussi M. G. & Mariotti M. A.: 1999a, 'Instruments for Perspective Drawing: Historic, Epistemological and Didactic Issues', in Goldschmidt G., Porter W. & Ozkar M. (Eds.), *Proc. of the 4th Int. Design Thinking Res. Symp. on Design Representation*, III 175-185, Massachusetts Institute of Technology & Technion - Israel Institute of Technology.

Bartolini Bussi M. G. & Mariotti M. A.: 1999b, 'Semiotic mediation : from history to mathematics classroom', *For the Learning of Mathematics,* 19 (2), 27-35.

Bartolini Bussi M. G.; Boni, M.; Ferri, F. & Garuti, R.: 1999c, 'Early Approach To Theoretical Thinking: Gears in Primary School', *Educ. Studies in Math.*, 39 (1-3), 67-87.

Berthoz, A.: 1997, *Le sens du mouvement*, Paris: Odile Jacob.

Boero, P.; Chiappini, G.P.; Garuti, R. & Sibilla, A.: 1995a, 'Towards Statements and Proof in Elementary Arithmetic', *Proc. of PME XIX*, Recife, vol. 3, 129-136.

Boero,P.; Dapueto, C.; Ferrari, P.; Ferrero, E.; Garuti ,R.; Lemut, E.; Parenti, L. & Scali, E.: 1995b, 'Aspects of the Mathematics-Culture Relationship in Mathematics Teaching-Learning in Compulsory School', in *Proc. PME XIX*, Recife, vol. 1, 151-166.

Boero, P.; Garuti, R.; Lemut, E.; Gazzolo, T. & Lladò, C.: 1995c, 'Some Aspects of the Construction of the Geometrical Conception of the Phenomenon of the Sun's Shadow', *Proc. PME XIX*, Recife, vol. 3, 11-18.

Boero, P.; Garuti, R.; Mariotti, M.A.: 1996, 'Some dynamic mental processes underlying producing and proving conjectures', *Proc. PME XX*, Valencia, vol. 2,121-128.

Dehaene, S. & Changeux, J.P: 1993, 'Development of Elementary Numerical Abilities: A Neuronal Model', *Journal of Cognitive Neuroscience*, 5, 390-407.

Dehaene, S.: 2000, *Il pallino della matematica*, Milano: Mondadori. (Italian translation from French of *La bosse de la mathématique*).

Douek, N. : 1999, 'Argumentative Aspects of Proving: Analysis of Some Undergraduate Mathematics Students' Performances', *Proc. PME XXIII*, Haifa, vol. 2, 273-280.

Gallistel, C.R. & Gelman, R.: 1992, 'Preverbal and verbal counting and computation', *Cognition*, 44, 43-74.

Garuti, R; Boero, P.; Lemut, E. & Mariotti, M.: 1996, 'Challenging the Traditional School Approach to Teorems: a hypothesis...', *Proc. PME XX*, Valencia, vol. 2, 113-120.

Garuti, R.; Boero, P. & Lemut, E.: 1998, 'Cognitive Unity of Theorems and Difficulty of Proof', *Proc. PME XXII*, Stellenbosch, vol. 2, 345-353.

Goldschmidt, G.: 1991, 'The Dialectics of Sketching', *Creativity research Journal*, 4, 2, 123-143.

Guala, E. & Boero, P.: 1999, 'Time Complexity and Learning', *Annals of the New York Academy of Sciences*, vol. 879, 164-167.

Harel, G. & Tall, D.: 1991, 'The general, the abstract, and the generic in advanced mathematics', *For the Learning of Mathematics*, 11 (1), 38-42.

Johnson, M.: 1987, *The body in the mind: the bodily basis of meaning, imagination and reason*, Chicago: The Univ. of Chicago Press.

Laborde, C.: 1999, 'The hidden role of diagrams in pupils' construction of meaning in geometry', preprint, to appear.

Lakoff,G. & Johnson, M: 1980, *The metaphors we live by*, Chicago: The Univ. of Chicago Press.

Lakoff, G. & Núñez, E.: 1996, 'The Metaphorical Structure of Mathematics: Sketching Out Cognitive Foundations For a Mind-Based Mathematics', in English, L. (Ed.), *Mathematical Reasoning Analogies, Metaphors and Images*, Hillsdale, NJ: Erlbaum.

Lolli, G.: 1999, 'Truth and Proofs', Lectures given at the *Winter School in the Epistemology of Mathematics for Didactics*, Levico Terme (Trento), 8-12 February 1999.

Longo, G.: 1997, 'Géometrie, Mouvement, Espace: Cognition et Mathématique', *Intellectica* (2), 25, 195-218. (it is a long review of Berthoz, 1997).

Love, E: 1995, 'The functions of visualisation in learning geometry', in: Sutherland, R. & Mason, J. (Ed.s): *Exploiting Mental Imagery with Computers in Mathematics Education*, NATO ASI Series, 138, Berlin: Springer, 125-141.

Mariotti, M.A. & Bartolini Bussi, M.G.: 1998, 'From Drawing to Construction: Teachers' mediation within the Cabri Environment', *Proc. PME XXII*, Stellenbosch, vol. 3, 247 254.

Mason, J. & Heal, B.: 1995, ' Mathematical screen metaphors', in Sutherland, R. & Mason, J. (Ed.s): *Exploiting Mental Imagery with Computers in Mathematics Education*, NATO ASI Series, 138, Berlin: Springer, 291-308.

Nemirovski, R: 1996, 'Mathematical Narratives, Modeling and Algebra', in N.Bednarz et al. (eds.), *Approaches to Algebra*, Amsterdam: Kluwer, 197-220.

Radford, L.: 1999, 'The Rhetoric of Generalization', *Proc. PME XXIII*, Haifa, vol. 4, 89-96.

Radford, L.: 2000, 'Semioticizing the General: Students' signs and Meanings in algebraic generalisation', preprint, to appear in *Educational Studies in Mathematics*.

Saada-Robert, M.: 1989, 'La microgénèse de la rapresentation d'un problème', *Psychologie Française*, 34, 2/3.

Simon, M: 1996, 'Beyond inductive and deductive reasoning', *Ed. Studies in Math.*, 30, 197-210.

Scrivener, S.A.R.: 1995, 'Exploiting mental imaging: reflections of an artist on a mathematical excursion', in: Sutherland, R. & Mason, J. (Ed.s): *Exploiting Mental Imagery with Computers in Mathematics Education*, NATO ASI Series, 138, Berlin: Springer, 309-321.

Sfard, A.: 1994, 'Reification as the Birth of Metaphor', *For the Learn. of Math.* 14 (1), 44-55.

Sfard, A.: 1997, 'Framing in Mathematical Discourse', *Proc. of PME-XXII*, Stellenbosch, vol. 4, 144-151.

Simone, R: 2000, *La terza fase*, Bari: Laterza.

Thurston, W.P: 1994, 'On Proof and Progress in Mathematics', *Bull. of the A.M.S.*, 30, 161-177.

Varela, F.J.: 1999, 'A Dimly Perceived Horizon: The Complex Meeting Ground between Physical and Inner Time', *Annals of the New York Academy of Sciences*, vol. 879, 143-153.

Vergnaud, G.: 1990, 'La théorie des champs conceptuels', *Recherches en didactiques des mathématiques*, 10, 133-169.

Vygotsky L.S.: 1934, *Myšlenie i reč (Thought and Language)*; also in: *Sobranie sočinenija* (1982-84); Italian translation from the two editions by L. Mecacci, Bari: Laterza, 1992.

Wertsch, J.V.: 1998, *Mind as Action*, New York and Oxford: Blackwell.

NOTICE

Reproduction Basis

- [x] This document is covered by a signed "Reproduction Release (Blanket)" form (on file within the ERIC system), encompassing all or classes of documents from its source organization and, therefore, does not require a "Specific Document" Release form.

- [ ] This document is Federally-funded, or carries its own permission to reproduce, or is otherwise in the public domain and, therefore, may be reproduced by ERIC without a signed Reproduction Release form (either "Specific Document" or "Blanket").